Potencias de la unidad imaginaria

Sea $n \in \mathbb{Z}^+ \cup \{0\}$. Entonces,

$$i^n = (-1)^{v_n}\big((1-u_n) + u_n i\big)$$

donde

$$u_n = \frac{1-(-1)^n}{2}, \quad v_n = \frac{n-u_n}{2}$$

Raíces de la unidad imaginaria

Sea $n \in \mathbb{Z}^+$. Si $i = (a+bi)^n$, con $a, b \in \mathbb{R}$, entonces:

$$\begin{cases} \displaystyle\sum_{k=0}^{v_n} (-1)^k \binom{n}{2k} a^{n-2k} b^{2k} = 0 \\ \displaystyle\sum_{k=0}^{v_{n-1}} (-1)^k \binom{n}{2k+1} a^{n-(2k+1)} b^{2k+1} = 1 \end{cases}$$

Es decir:

- ✓ $b \neq 0$.
- ✓ Si $a \neq 0$, entonces,

$$a^n = \frac{1}{\sum_{k=0}^{v_{n-1}} (-1)^k \binom{n}{2k+1} \left(\frac{b}{a}\right)^{2k+1}}$$

, donde,

$$\sum_{k=0}^{v_n} (-1)^k \binom{n}{2k} \left(\frac{b}{a}\right)^{2k} = 0$$

, o, $a = 0$ si y sólo si n es impar y $b = (-1)^{\frac{n-1}{2}}$.

Raíces de la unidad real

Sea $n \in \mathbb{Z}^+$. Si $1 = (a+bi)^n$, con $a, b \in \mathbb{R}$, entonces:

$$\begin{cases} \displaystyle\sum_{k=0}^{v_n} (-1)^k \binom{n}{2k} a^{n-2k} b^{2k} = 1 \\ \displaystyle\sum_{k=0}^{v_{n-1}} (-1)^k \binom{n}{2k+1} a^{n-(2k+1)} b^{2k+1} = 0 \end{cases}$$

Es decir:

- ✓ $a \neq 0$.
- ✓ Si $b \neq 0$, entonces,

$$a^n = \frac{1}{\sum_{k=0}^{v_n} (-1)^k \binom{n}{2k} \left(\frac{b}{a}\right)^{2k}}$$

, donde,

$$\sum_{k=0}^{v_{n-1}} (-1)^k \binom{n}{2k+1} \left(\frac{b}{a}\right)^{2k} = 0$$

, o, $b = 0$ si y sólo si $a^n = 1$.

Por otro lado, para $x \in \mathbb{R}$,

$$\frac{d^n}{dx^n}(\cos x) = (-1)^{v_n}\big((1-u_n)\cos x - u_n \sin x\big)$$

$$\frac{d^n}{dx^n}(\sin x) = (-1)^{v_n}(u_n \cos x + (1-u_n)\sin x)$$

Ahora, haciendo uso de los polinomios de Taylor centrados en cero, se obtiene:

$$\cos x = \sum_{n=0}^{\infty} (-1)^{v_n}(1-u_n)\frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} (-1)^{v_n} u_n \frac{x^n}{n!}$$

Luego,

$$\cos x + i \sin x = \sum_{n=0}^{\infty} \frac{x^n i^n}{n!} = e^{xi}$$

Ahora,

$$\left(e^{xi}\right)^n = (\cos x + i \sin x)^n = \sum_{k=0}^{n} \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k \qquad (0)$$

Pero,

$$e^{(nx)i} = \cos nx + i \sin nx$$

Por consiguiente:

Fórmulas de ángulo múltiple para el coseno

$$\cos nx = \sum_{k=0}^{v_n} \binom{n}{2k} (-1)^k (\cos x)^{n-2k} (\sin x)^{2k}$$

$$= \sum_{k=0}^{v_n} \binom{n}{2k} \left(\sum_{p=0}^{k} \binom{k}{p} (-1)^p (\cos x)^{n-2p} \right)$$

$$= \sum_{k=0}^{v_n} \left(\sum_{p=k}^{v_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \qquad (1)$$

Observación:

$$\sum_{k=0}^{n} a_k \left(\sum_{p=0}^{k} b_p c_{p,k} \right) = \sum_{k=0}^{n} \left(\sum_{p=k}^{n} a_p c_{k,p} \right) b_k$$

Luego,

$$\sum_{n=0}^{m}\cos nx=\sum_{n=0}^{m}\left(\sum_{k=0}^{v_{m-u_n}-v_n}\left(\sum_{p=k}^{v_{2k+n}}\binom{2k+n}{2p}\binom{p}{k}\right)(-1)^k\right)(\cos x)^n=\frac{\sin(m+1)\frac{x}{2}}{\sin\frac{x}{2}}\cos m\frac{x}{2}\qquad(2)$$

Observación:

$$\sum_{n=0}^{m}\left(\sum_{k=0}^{v_n}a_k b_{n-2k}c_{n,k}\right)=\left[\sum_{n=0}^{v_m}\left(\sum_{k=0}^{v_{m-n}}a_k c_{2k+2n,k}\right)b_{2n}\right]+\left[\sum_{n=0}^{v_{m-1}}\left(\sum_{k=0}^{v_{m-1-n}}a_k c_{2k+(2n+1),k}\right)b_{2n+1}\right]=\sum_{n=0}^{m}\left(\sum_{k=0}^{v_{m-a_n}-v_n}a_k c_{2k+n,k}\right)b_n$$

Consideremos de (1):

$$\cos nx=(\cos x)^n\sum_{p=0}^{v_n}\binom{n}{2p}+\sum_{k=1}^{v_n}\left(\sum_{p=k}^{v_n}\binom{n}{2p}\binom{p}{k}\right)(-1)^k(\cos x)^{n-2k}$$

Por consiguiente,

$$(\cos x)^n=\frac{1}{2^{n-1}}\left(\cos nx-\sum_{k=1}^{v_n}\left(\sum_{p=k}^{v_n}\binom{n}{2p}\binom{p}{k}\right)(-1)^k(\cos x)^{n-2k}\right)$$

Observación:

$$\sum_{p=0}^{v_n}\binom{n}{2p}=2^{n-1}$$

Ahora, usando (2) se obtiene:

$$\sum_{m=0}^{q}(\cos mx)^n=\frac{1}{2^{n-1}}\left(\frac{\sin(q+1)\frac{nx}{2}}{\sin\frac{nx}{2}}\cos q\frac{nx}{2}-\sum_{m=0}^{q}\sum_{k=1}^{v_n}(-1)^k\left(\sum_{p=k}^{v_n}\binom{n}{2p}\binom{p}{k}\right)(\cos mx)^{n-2k}\right)$$

$$=\frac{1}{2^{n-1}}\left(\frac{\sin(q+1)\frac{nx}{2}}{\sin\frac{nx}{2}}\cos q\frac{nx}{2}-\sum_{k=1}^{v_n}(-1)^k\left(\sum_{p=k}^{v_n}\binom{n}{2p}\binom{p}{k}\right)\left(\sum_{m=0}^{q}(\cos mx)^{n-2k}\right)\right)\qquad(*)$$

Observación:

$$\sum_{m=0}^{q}\left(\sum_{k=1}^{n}a_k b_{m,k}\right)=\sum_{k=1}^{n}a_k\left(\sum_{m=0}^{q}b_{m,k}\right)$$

Fórmulas de ángulo múltiple para el seno

$$\sin nx=\sum_{k=0}^{v_{n-1}}(-1)^k\binom{n}{2k+1}(\cos x)^{n-(2k+1)}(\sin x)^{2k+1}$$

$$=(\cos x)^{u_{n-1}}\sum_{k=0}^{v_{n-1}}\binom{n}{2k+1}\left(\sum_{p=k}^{v_{n-1}}(-1)^p\binom{v_{n-1}-k}{v_{n-1}-p}(\sin x)^{2p+1}\right)$$

$$= (\cos x)^{u_{n-1}} \sum_{k=0}^{v_{n-1}} \left(\sum_{p=0}^{k} \binom{n}{2p+1} \binom{v_{n-1}-p}{v_{n-1}-k} \right) (-1)^k (\sin x)^{2k+1} \qquad (3)$$

Observación:

$$\sum_{k=0}^{n} a_k \left(\sum_{p=k}^{n} b_p c_{p,k} \right) = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} a_p c_{k,p} \right) b_k$$

Luego,

$$\sum_{n=0}^{m} \sin nx = \sum_{n=0}^{v_{m-1}} (-1)^n (\sin x)^{2n+1} \left(\sum_{k=2n+1}^{m} (\cos x)^{u_{n-1}} \left(\sum_{p=0}^{n} \binom{k}{2p+1} \binom{v_{k-1}-p}{v_{k-1}-n} \right) \right) = \frac{\sin(m+1)\frac{x}{2}}{\sin\frac{x}{2}} \sin m\frac{x}{2}$$

Observación:

$$\sum_{n=0}^{m} a_n \left(\sum_{k=0}^{v_{n-1}} b_k c_{n,k} \right) = \sum_{n=0}^{v_{m-1}} \left(\sum_{k=2n+1}^{m} a_k c_{k,n} \right) b_n \ \wedge \ a_0 = 0$$

Consideremos de (3):

$$\sin(2n+1)x = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \qquad (4)$$

Si $x = \frac{\pi}{2n+1}$, se obtiene:

$$\sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k \left(\sin\frac{\pi}{2n+1} \right)^{2k} = 0$$

Si $n = 2$, se obtiene la expresión $16\left(\sin\frac{\pi}{5}\right)^4 - 20\left(\sin\frac{\pi}{5}\right)^2 + 5 = 0$. Puesto que $\sin\frac{\pi}{6} < \sin\frac{\pi}{5} < \sin\frac{\pi}{4}$, entonces $\sin\frac{\pi}{5} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$.

Por otro lado, de (4):

$$\sum_{n=0}^{v_m} \sin(2n+1)x = \sum_{n=0}^{v_m} \left(\sum_{k=n}^{v_m} \left(\sum_{p=0}^{n} \binom{2k+1}{2p+1} \binom{k-p}{k-n} \right) \right) (-1)^n (\sin x)^{2n+1} = \frac{(\sin v_m x)^2}{\sin x} \qquad (5)$$

Observación:

$$\sum_{n=0}^{m} \left(\sum_{k=0}^{n} a_k b_{n,k} \right) = \sum_{n=0}^{m} \left(\sum_{k=n}^{m} b_{k,n} \right) a_n$$

Consideremos de (4):

$$\sin(2n+1)x = (-1)^n (\sin x)^{2n+1} \sum_{p=0}^{n} \binom{2n+1}{2p+1} + \sum_{k=0}^{n-1} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1}$$

Por consiguiente,

$$(\sin x)^{2n+1}=\frac{(-1)^n}{2^{2n}}\left(\sin(2n+1)x-\sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin x)^{2k+1}\right)$$

Observación:

$$\sum_{p=0}^{n}\binom{2n+1}{2p+1}=2^{2n}$$

Ahora, usando (5) se obtiene:

$$\sum_{m=0}^{q}(\sin mx)^{2n+1}=\frac{(-1)^n}{2^{2n}}\left(\frac{\sin(q+1)\frac{(2n+1)x}{2}}{\sin\frac{(2n+1)x}{2}}\sin q\,\frac{(2n+1)x}{2}-\sum_{m=0}^{q}\left(\sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin mx)^{2k+1}\right)\right)$$

$$=\frac{(-1)^n}{2^{2n}}\left(\frac{\sin(q+1)\frac{(2n+1)x}{2}}{\sin\frac{(2n+1)x}{2}}\sin q\,\frac{(2n+1)x}{2}-\sum_{k=0}^{n-1}(-1)^k\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)\left(\sum_{m=0}^{q}(\sin mx)^{2k+1}\right)\right)\quad(**)$$

Observación:

$$\sum_{m=0}^{q}\left(\sum_{k=0}^{n}a_k b_{m,k}\right)=\sum_{k=0}^{n}a_k\left(\sum_{m=0}^{q}b_{m,k}\right)\ \wedge\ b_{0,k}=0$$

Finalmente, hallamos que las expresiones $(*)$ y $(**)$ son similares a la recurrencia entre coeficientes de la fórmula polinómica general de la suma de las n-ésimas y $2n+1$-ésimas potencias, respectivamente, de los primeros q naturales.